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On the stability of elastic annular rods

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Abstract

The stability of equilibrium of non-linearly elastic rods, whose deformations obey the classical Kirchhoff's equations, is considered. A variational formulation of the equilibrium problem is given, and the equilibrium equations for infinitesimal deformations superimposed to a finite transformation of a rod are deduced. The stability of annular rings, in which the twisting strain is non-null, is investigated by study of the second variation of the energy functional.

Keywords: Elastic stability; Non-linear elasticity; Rod theory

1. Introduction

In this paper we examine the problem of stability of equilibrium configurations of elastic rods in the context of a theory that is a generalization to non-linear constitutive equations of the classical Kirchhoff's theory of inextensible rods. According to a general concept of static stability, an equilibrium configuration of an hyperelastic body is stable when it realizes a minimum of the energy, and, hence, the stability of a state of equilibrium is usually examined by an analysis of the second variation of the energy functional (cf. Coleman and Noll, 1959; Truesdell and Noll, 1965, Sections 68bis, 89; Wang and Truesdell, 1973, Chapter VII; Iooss and Joseph, 1997, Chapter XII). In the present paper, in view of obtaining the tools for a discussion on the stability of equilibrium of inextensible rods, we give a characterization of the equilibrium equations as the extremals of a functional $\mathscr{J}$, which is obtained by adding to the energy functional a term that accounts for the presence of the kinematical constraint expressing the assumption that the rod axis is inextensible. Then we deduce the form that $\delta^2\mathscr{J}$, the second variation of $\mathscr{J}$, takes on an extremal, so that we can investigate the stability of an equilibrium state by checking the sign of $\delta^2\mathscr{J}$. In the perspective of using, to test the sign of $\delta^2\mathscr{J}$, functions having the same form as the eigenfunctions of the linearized equilibrium problem, we also deduce the equilibrium equations for infinitesimal deformations superimposed to a finite transformation of a rod from a natural state to a state of equilibrium. As an application of the derived equations, we discuss the stability of annular rings for which the measure μ_3 of twisting strain with respect to a natural state is non-null. These circular equilibrium configurations can be thought of as having been

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obtained by joining and sealing the two ends of rods that are open in a natural state. We consider circular rings formed from materially homogeneous inextensible rods obeying Kirchhoff's linear constitutive equation and whose axes are curves of uniform curvature and torsion in a natural state. Annular equilibrium configurations with $\mu_3 \neq 0$ are possible for two classes of rods of this type. The rods of the first class have the shape of an helix when undeformed and, for them, the twisting strain μ_3 in the annular configuration equals the opposite of the geometric torsion of the axial curve in a natural state. The rods in the second class are prismatic in a natural state and are deformed in a circular ring after the imposition of an amount κ_3 of uniform twist density. For these rods, the twisting strain μ_3 in the circular equilibrium configuration equals the twist density κ_3. We shall call the rods of the two classes *naturally helicoidal* and *naturally straight*, respectively. It is known that, for a naturally straight rod, there is a critical amount of the absolute value of the twist density $\kappa_3 = \mu_3$ in the circular configuration, and that, for $|\kappa_3|$ greater than the critical value, the annular configuration (in which the axial curve is a circle) is unstable and the rod tends to buckle into a configuration in which the axial curve is not planar. We examine in detail the problem of stability of annular rings formed from naturally helicoidal rods. We find that the equations governing the stability problem for this class of rods depend on two dimensionless parameters which are defined in terms of the (invariable) length of the axial curve and its curvature and torsion in a natural state. The values of these two parameters for which the equations of the linearized equilibrium problem have non-trivial solutions, define a family of eigencurves. The study of the sign of $\delta^2\mathscr{J}$ shows that this second variation vanishes on the eigencurves and that there is a loss of stability of the annular configuration in crossing the first eigencurve. The results of the discussion on the stability furnish the values of the curvature, torsion, and length of the undeformed axial curve for which the corresponding annular equilibrium configuration is stable.

The problem of stability of elastic rods has attracted the attention of the researchers since Euler's discovery of the phenomenon of buckling (1743) and the first investigations of Euler, Daniel and John Bernoulli, and Lagrange on the load that may cause the bending of a compressed column (cf. Truesdell, 1960; Timoshenko, 1983). The literature concerning the stability of rods is voluminous and continuously enriched by new studies that elucidate previously unknown aspects of the problem or present new applications of existing theories. Among recent contributions to the subject, we recall Green et al. (1968), Antman (1984), Caflish and Maddocks (1984), Maddocks (1984), and refer the reader to Antman (1995) for a modern presentation of the matter including recent results and an extensive bibliography. The interest for the problem here considered of the stability of annular rods, besides the possible mechanical applications, is related to the use of rod theory to construct an elastic model for the DNA molecule (cf. e.g., Coleman et al., 1995; Swigon et al., 1998). A segment of DNA is usually represented as a double helix in which two strands wind around a common axis that may form a closed and/or not planar curve. In applications of the elasticity theory to the study of DNA, the molecule is usually viewed as an homogeneous inextensible elastic rod with uniform cross-sections; a configuration of the rod specifies a configuration of the DNA molecule because it gives the position in space of the duplex axis and the rate of twisting of the strands about the axis. Recent studies consider the behavior of molecules of DNA in configurations that are not relaxed (natural) states. This is the case, for instance, of the paper (Tobias et al., 1996), where some finite motions of pure torsion possible for closed circular loops of DNA are presented, and the difference in behavior between loops formed from naturally straight rods and loops formed from rods that possess intrinsic uniform curvature and torsion are discussed. In cases of this type, that deal with states of equilibrium that are not stress-free, the knowledge of the properties of stability of equilibrium configurations may be of interest. Among recent works that employ the elastic rod model for the study of stability of DNA configurations, we recall Tobias et al. (2000) and Coleman et al. (2000).

The present paper is organized as follows: Section 2 gives a presentation of the rod theory on which the subsequent analysis is based; Section 3 contains a formulation of the equilibrium equations of rods as the extremals of a functional $\mathscr{J}$ and includes a deduction of the second variation of this functional; in Section 4 the equilibrium equations for infinitesimal deformations from an arbitrary equilibrium configuration of the rod are derived; finally, in Section 5, as an application of the results of the preceding sections, the stability of annular rings with non-null twisting strain is discussed. In the paper the summation convention is adopted, with the agreement that Latin indices range over $\{1,2,3\}$ and Greek indices over $\{1,2\}$. The symbol $\otimes$ is used to denote the tensor product $\boldsymbol{u}\otimes\boldsymbol{v}$ of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, that assigns to each vector $\boldsymbol{a}$ the vector $(\boldsymbol{a}\cdot\boldsymbol{v})\boldsymbol{u}$.

2. Equilibrium equations

In the rod theory here considered the kinematics is the same as that of Kirchhoff's non-linear theory of inextensible rods (cf. Love, 1944; Dill, 1992; Coleman et al., 1992), but the constitutive equation for the moment vector is more general, because it is assumed only that the rod is hyperelastic without imposing a linear dependence of the moment upon the strain measure.

A rod $\mathscr{R}$ is regarded as a three-dimensional body possessing an undistorted stress-free configuration $\mathfrak{C}^{\mathrm{u}}$ that can be described as follows: Let $\mathscr{C}^{\mathrm{u}}$ be a smooth space curve defined by the equation $\boldsymbol{x}^{\mathrm{u}}=\boldsymbol{x}^{\mathrm{u}}(s)$, with s an arc-length parameter that varies over the interval $[0,L]$. In the configuration $\mathfrak{C}^{\mathrm{u}}$, the positions $\hat{\boldsymbol{x}}^{\mathrm{u}}$ of the points of $\mathscr{R}$ are given by

$$\hat{\boldsymbol{x}}^{\mathrm{u}}(X_1,X_2,s)=\boldsymbol{x}^{\mathrm{u}}(s)+X_1\boldsymbol{d}_1^{\mathrm{u}}(s)+X_2\boldsymbol{d}_2^{\mathrm{u}}(s), \tag{1}$$

where the pairs (X_1,X_2) belong to a connected plane region, and $\boldsymbol{d}_1^{\mathrm{u}}(s)$ and $\boldsymbol{d}_2^{\mathrm{u}}(s)$ are unit vectors orthogonal to $\mathscr{C}^{\mathrm{u}}$ at $\boldsymbol{x}^{\mathrm{u}}(s)$ and such that the triplet $(\boldsymbol{d}_1^{\mathrm{u}}(s),\boldsymbol{d}_2^{\mathrm{u}}(s),\boldsymbol{d}_3^{\mathrm{u}}(s))$, with $\boldsymbol{d}_3^{\mathrm{u}}(s)=\mathrm{d}\boldsymbol{x}^{\mathrm{u}}(s)/\mathrm{d}s$, is right-handed and orthonormal. The cross-sections $\mathscr{S}(s)$ of $\mathscr{R}$, i.e., the sets of points for which s is constant in (1), are assumed to be congruent and such that they contain their centroids; the curve $\mathscr{C}^{\mathrm{u}}$ is taken to be the locus of these centroids. The material points that are on $\mathscr{C}^{\mathrm{u}}$ form the *axis* of $\mathscr{R}$, and $\mathscr{C}^{\mathrm{u}}$ is called the *axial curve* of $\mathscr{R}$ in the configuration $\mathfrak{C}^{\mathrm{u}}$. For each s, the vectors $\boldsymbol{d}_1^{\mathrm{u}}(s)$ and $\boldsymbol{d}_2^{\mathrm{u}}(s)$ are chosen to have the directions of the principal axes of inertia of $\mathscr{S}(s)$. When the three-dimensional body $\mathscr{R}$ is called a rod, it is presupposed that the maximal distance between two points of a cross-section is small compared to the length L of the axial curve. A rod of the type here considered is *inextensible* in the sense that, in its deformations, the change in arc-length distance between points of the axis can be disregarded.

In each configuration $\mathfrak{C}$, the points of the axis form a curve $\mathscr{C}$, the axial curve of $\mathscr{R}$ in $\mathfrak{C}$, which is described by an equation of the form $\boldsymbol{x}=\boldsymbol{x}(s)$. Since the rod inextensible, s is an arc-length parameter for $\mathscr{C}$ and the derivative of $\boldsymbol{x}$ with respect to s is the unit tangent $\boldsymbol{t}(s)$ at the point $\boldsymbol{x}(s)$ of $\mathscr{C}$:

$$\boldsymbol{t}(s)=\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{x}(s)=\boldsymbol{x}_{,s}(s). \tag{2}$$

At each point $\boldsymbol{x}(s)$ in the configuration $\mathfrak{C}$, let $\boldsymbol{d}_1(s)$ and $\boldsymbol{d}_2(s)$ be the unit vectors having the directions of the material fibers that lie along $\boldsymbol{d}_1^{\mathrm{u}}(s)$ and $\boldsymbol{d}_2^{\mathrm{u}}(s)$, at $\boldsymbol{x}^{\mathrm{u}}(s)$ in the configuration $\mathfrak{C}^{\mathrm{u}}$. The vectors $\boldsymbol{d}_1(s)$ and $\boldsymbol{d}_2(s)$ can be regarded as orthogonal to each other and to the curve $\mathscr{C}$ (cf. Love, 1944, Art. 252; Dill, 1992). The configuration $\mathfrak{C}$ is determined by giving: (i) the equation $\boldsymbol{x}=\boldsymbol{x}(s)$ of the axial curve $\mathscr{C}$, and (ii) the vector-valued function $\boldsymbol{d}_1=\boldsymbol{d}_1(s)$. Then, at each point $\boldsymbol{x}(s)$ of $\mathscr{C}$ can be associated a triad $(\boldsymbol{d}_1(s),\boldsymbol{d}_2(s),\boldsymbol{d}_3(s))$ of orthonormal vectors, with $\boldsymbol{d}_3(s)=\boldsymbol{t}(s)$ and $\boldsymbol{d}_2(s)=\boldsymbol{d}_3(s)\times\boldsymbol{d}_1(s)$. Since the triplets $(\boldsymbol{d}_1,\boldsymbol{d}_2,\boldsymbol{d}_3)$ are formed by orthogonal unit vectors, there exists a vector

$$\boldsymbol{\kappa}(s)=\kappa_i(s)\boldsymbol{d}_i(s)=\frac{1}{2}\boldsymbol{d}_i(s)\times\boldsymbol{d}_{i,s}(s), \tag{3}$$

called the *curvature vector* for the configuration $\mathfrak{C}$, such that

$$\boldsymbol{d}_{i,s}(s)=\boldsymbol{\kappa}(s)\times\boldsymbol{d}_i(s),\quad i=1,2,3. \tag{4}$$

The components $\kappa_\alpha = \boldsymbol{\kappa} \cdot \boldsymbol{d}_\alpha$ of $\boldsymbol{\kappa}$ are called *components of curvature* and give the geometric curvature k of the axial curve through the formula $k = \sqrt{\kappa_\alpha \kappa_\alpha}$; the component $\kappa_3 = \boldsymbol{\kappa} \cdot \boldsymbol{d}_3$, along the tangent to $\mathscr{C}$, gives the *twist density* of $\mathscr{R}$ in the configuration $\mathfrak{C}$. The unit vectors $\boldsymbol{d}_1(s)$ and $\boldsymbol{d}_2(s)$ lie in the plane of, but are in general distinct from, the principal normal $\boldsymbol{n}(s)$ and the binormal $\boldsymbol{b}(s)$ of $\mathscr{C}$ at $\boldsymbol{x}(s)$. We denote by $\varphi(s)$ the angle from $\boldsymbol{n}(s)$ to $\boldsymbol{d}_1(s)$. The twist density κ_3 of $\mathfrak{C}$ and the geometric torsion τ of $\mathscr{C}$ are related by

$$\kappa_3 = \tau + \varphi_{,s}. \tag{5}$$

Let $\boldsymbol{\kappa}^{\mathrm{u}}$ be the curvature vector in the undistorted stress-free configuration $\mathfrak{C}^{\mathrm{u}}$, and let $\boldsymbol{R}$ be the rotation that transforms the triplets $(\boldsymbol{d}_1^{\mathrm{u}}, \boldsymbol{d}_2^{\mathrm{u}}, \boldsymbol{d}_3^{\mathrm{u}})$ into the triplets $(\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3)$:

$$\boldsymbol{R}(s) = \boldsymbol{d}_i(s) \otimes \boldsymbol{d}_i^{\mathrm{u}}(s), \qquad \boldsymbol{d}_i(s) = \boldsymbol{R}(s)\boldsymbol{d}_i^{\mathrm{u}}(s), \quad i = 1, 2, 3. \tag{6}$$

The vector

$$\boldsymbol{\mu} = \boldsymbol{\kappa} - \boldsymbol{R}\boldsymbol{\kappa}^{\mathrm{u}} = \frac{1}{2}\boldsymbol{R}\boldsymbol{d}_i^{\mathrm{u}} \times \boldsymbol{R}_{,s}\boldsymbol{d}_i^{\mathrm{u}}, \tag{7}$$

is the strain measure in the deformation that transforms the rod from $\mathfrak{C}^{\mathrm{u}}$ to $\mathfrak{C}$; the components of $\boldsymbol{\mu}$ are:

$$\mu_i = \boldsymbol{\mu} \cdot \boldsymbol{d}_i = \kappa_i - \kappa_i^{\mathrm{u}}, \quad i = 1, 2, 3, \tag{8}$$

with $\kappa_i^{\mathrm{u}} = \boldsymbol{\kappa}^{\mathrm{u}} \cdot \boldsymbol{d}_i^{\mathrm{u}}$. The first two components of $\boldsymbol{\mu}$ give a measure of the flexural deformation of $\mathscr{R}$; the third component measures the *twisting strain*, given by the change in twist density.

When the only external loads acting on the rod are forces and couples applied to the ends, in an equilibrium configuration $\mathfrak{C}$ the following equations hold:

$$\boldsymbol{f}_{,s} = \boldsymbol{0}, \quad \boldsymbol{m}_{,s} + \boldsymbol{d}_3 \times \boldsymbol{f} = \boldsymbol{0}, \tag{9}$$

where $\boldsymbol{f} = \boldsymbol{f}(s)$, the force vector, is the resultant of the stresses acting on the cross-section at s, and $\boldsymbol{m} = \boldsymbol{m}(s)$, the moment vector, is the resultant moment of these stresses about the centroid of the section. The force $\boldsymbol{f}$ is a reactive variable that is not constitutively determined; the moment $\boldsymbol{m}$ is obtained from the strain measure through a constitutive equation. We assume that the rod is hyperelastic, with a strain energy density per unit length $\sigma = \sigma(\mu_1, \mu_2, \mu_3; s)$ such that

$$\boldsymbol{m} = m_i\boldsymbol{d}_i, \quad m_i = \boldsymbol{m} \cdot \boldsymbol{d}_i = \frac{\partial\sigma}{\partial\mu_i}, \quad i = 1, 2, 3. \tag{10}$$

The components m_1 and m_2 of $\boldsymbol{m}$ are the *bending moments*, and the component m_3 is the *twisting moment*. In addition to Eq. (9), with $\boldsymbol{m}$ given by (10), in an equilibrium configuration must hold the equation

$$\boldsymbol{d}_3 = \boldsymbol{x}_{,s}, \tag{11}$$

which follows from the assumption that the rod axis is inextensible. We shall refer to this constraint as the *condition of inextensibility*.

When $\mathfrak{C}^{\mathrm{u}}$ is assumed as reference configuration, a deformation of $\mathscr{R}$ can be described by specification of the displacement $\boldsymbol{u}$ of the points of the axis,

$$\boldsymbol{u}(s) = \boldsymbol{x}(s) - \boldsymbol{x}^{\mathrm{u}}(s), \tag{12}$$

and the rotation $\boldsymbol{R}$ defined by $(6)_1$. Thus a deformation of $\mathscr{R}$ is determined by six scalar functions: the three components of $\boldsymbol{u}$ and three parameters defining $\boldsymbol{R}$, which are usually chosen to be Euler angles. These six functions cannot be assigned arbitrarily because, by $(6)_2$, (11), and (12), they must be such that

$$(\boldsymbol{R} - \boldsymbol{I})\boldsymbol{d}_3^{\mathrm{u}} = \boldsymbol{u}_{,s}, \tag{13}$$

where $\boldsymbol{I}$ is the unit tensor. Eq. (13) expresses the condition of inextensibility in terms of $\boldsymbol{R}$ and $\boldsymbol{u}$. In view of (6), (7) and (10), the equilibrium equations (9) and the constraint equation (13) form a system of three (vectorial) equations for the three unknowns $\boldsymbol{u}$, $\boldsymbol{R}$, and $\boldsymbol{f}$.

In the following two sections, that treat the variational formulation and the linearization of the equilibrium equations, we do not specify the form of the strain energy density, and in particular we do not assume that the components of $\boldsymbol{m}$ are linear functions of the components of $\boldsymbol{\mu}$. In the special case in which this assumption is made and the rod is composed of an isotropic material, Eqs. $(10)_2$ reduce to the classical Kirchhoff's linear constitutive relations in which

$$\sigma = \frac{1}{2}(A_1(\mu_1)^2 + A_2(\mu_2)^2 + C(\mu_3)^2), \tag{14}$$

and, hence,

$$\begin{aligned} m_\alpha &= A_\alpha \mu_\alpha = A_\alpha(\kappa_\alpha - \kappa_\alpha^{\mathrm{u}}), \quad \alpha = 1,2, \quad \text{(no sum)} \\ m_3 &= C\mu_3 = C(\kappa_3 - \kappa_3^{\mathrm{u}}), \end{aligned} \tag{15}$$

where A_1 and A_2 are the flexural rigidities and C is the torsional rigidity of the cross-sections. We shall adopt a constitutive equation of the type (15) in Section 5 where the stability of annular rods is discussed.

3. Variational formulation

In this section we formulate a variational principle that characterizes the equilibrium configurations of an hyperelastic rod $\mathscr{R}$ as the extremals of a functional $\mathscr{J}$ in which the force vector appears as a Lagrange multiplier associated with the kinematical constraint (13). In view of applying our results to investigations on the stability of equilibrium states, we also deduce the expression that the second variation of $\mathscr{J}$ assumes on an extremal, that is in correspondence of an equilibrium configuration of $\mathscr{R}$.

Let $\mathfrak{C}$ be an equilibrium configuration in which the rod $\mathscr{R}$ has been transformed from a stress-free reference configuration $\mathfrak{C}^{\mathrm{u}}$. The configuration $\mathfrak{C}$ is determined by the displacement $\boldsymbol{u}$ of the points of the axis from the curve $\mathscr{C}^{\mathrm{u}}$, and the rotation $\boldsymbol{R}$ that transforms the vectors $\boldsymbol{d}_i^{\mathrm{u}}$ of $\mathfrak{C}^{\mathrm{u}}$ into the vectors $\boldsymbol{d}_i$ of $\mathfrak{C}$. A variation in the configuration $\mathfrak{C}$ is characterized by a variation $\delta\boldsymbol{u}$ of $\boldsymbol{u}$ and a variation $\delta\boldsymbol{R}$ of $\boldsymbol{R}$. As a consequence of the condition of inextensibility (13), the functions $\delta\boldsymbol{u}$ and $\delta\boldsymbol{R}$ are not independent.

We observe that the variation $\delta\boldsymbol{R}$ determines the variations $\delta\boldsymbol{d}_i$ of the vectors $\boldsymbol{d}_i$ and vice versa. Namely, it follows from Eqs. $(6)_2$ that in correspondence of a variation $\delta\boldsymbol{R}$ the vectors $\boldsymbol{d}_i$ undergo the variations

$$\delta\boldsymbol{d}_i = (\delta\boldsymbol{R})\boldsymbol{d}_i^{\mathrm{u}} = (\delta\boldsymbol{R})\boldsymbol{R}^{\mathrm{T}}\boldsymbol{d}_i, \quad i = 1,2,3. \tag{16}$$

Let $\delta\boldsymbol{q}$ denote the axial vector corresponding to the skew-tensor $(\delta\boldsymbol{R})\boldsymbol{R}^{\mathrm{T}}$, i.e., the vector $\delta\boldsymbol{q}$ such that $\delta\boldsymbol{q} \times \boldsymbol{a} = (\delta\boldsymbol{R})\boldsymbol{R}^{\mathrm{T}}\boldsymbol{a}$ for each vector $\boldsymbol{a}$. The relationship between $\delta\boldsymbol{q}$ and $\delta\boldsymbol{R}$ can be expressed by the formulae

$$\delta\boldsymbol{q} = \frac{1}{2}\boldsymbol{d}_i \times (\delta\boldsymbol{R})\boldsymbol{R}^{\mathrm{T}}\boldsymbol{d}_i, \quad \delta\boldsymbol{R} = \boldsymbol{d}_i \otimes \boldsymbol{R}^{\mathrm{T}}(\boldsymbol{d}_i \times \delta\boldsymbol{q}), \tag{17}$$

and, in terms of the vector $\delta\boldsymbol{q}$, the variations of the vectors $\boldsymbol{d}_i$ can be written as

$$\delta\boldsymbol{d}_i = \delta\boldsymbol{q} \times \boldsymbol{d}_i, \quad i = 1,2,3. \tag{18}$$

Eqs. (16)–(18) imply that

$$\delta\boldsymbol{q} = \frac{1}{2}\boldsymbol{d}_i \times \delta\boldsymbol{d}_i, \quad \delta\boldsymbol{R} = -\boldsymbol{d}_i \otimes \boldsymbol{R}^{\mathrm{T}}(\delta\boldsymbol{d}_i), \tag{19}$$

and thus that the variations $\delta\boldsymbol{d}_i$ determine $\delta\boldsymbol{q}$ and $\delta\boldsymbol{R}$. Eqs. (3), (7), (17), and (18) yield that the variation of the strain measure $\boldsymbol{\mu}$ has the form

$$\delta\boldsymbol{\mu} = \delta\boldsymbol{q}_{,s} + \delta\boldsymbol{q} \times \boldsymbol{\mu}. \tag{20}$$

From this equation, taking (18) into account, we have the following expressions for the variations of the components of $\boldsymbol{\mu}$:

$$\delta\mu_i = \delta(\boldsymbol{\mu} \cdot \boldsymbol{d}_i) = \delta\boldsymbol{q}_{,s} \cdot \boldsymbol{d}_i, \quad i = 1, 2, 3. \tag{21}$$

Then from the constitutive equation (10), in view of (18) and (21), we deduce that

$$\delta\sigma = \frac{\partial\sigma}{\partial\mu_i}\delta\mu_i = \boldsymbol{m} \cdot \delta\boldsymbol{q}_{,s}, \tag{22}$$

and

$$\delta\boldsymbol{m} = \delta\left(\frac{\partial\sigma}{\partial\mu_i}\boldsymbol{d}_i\right) = \boldsymbol{C}\delta\boldsymbol{q}_{,s} + \delta\boldsymbol{q} \times \boldsymbol{m}, \tag{23}$$

where the second-order tensor $\boldsymbol{C}$ is defined by

$$\boldsymbol{C} = \frac{\partial^2\sigma}{\partial\mu_i\partial\mu_j}\boldsymbol{d}_i \otimes \boldsymbol{d}_j. \tag{24}$$

Eq. (22) yields that the variation of the strain energy $\mathscr{U}$ of $\mathscr{R}$ can be written as

$$\delta\mathscr{U} = \delta\int_0^L \sigma\,\mathrm{d}s = -\int_0^L \boldsymbol{m}_{,s} \cdot \delta\boldsymbol{q}\,\mathrm{d}s + [\boldsymbol{m} \cdot \delta\boldsymbol{q}]_0^L. \tag{25}$$

We denote by $\mathscr{W}$ the work done by the external forces $\hat{\boldsymbol{f}}$ and couples $\widehat{\boldsymbol{m}}$ acting on the end sections of $\mathscr{R}$, and consider the functional

$$\mathscr{J} = \mathscr{U} - \mathscr{W} + \mathscr{L}, \tag{26}$$

obtained by adding to the energy associated with a deformation of $\mathscr{R}$ the term

$$\mathscr{L} = \int_0^L \boldsymbol{\xi} \cdot (\boldsymbol{u}_{,s} - (\boldsymbol{R} - \boldsymbol{I})\boldsymbol{d}_3^{\mathrm{u}})\,\mathrm{d}s, \tag{27}$$

which accounts for the condition of inextensibility (13), and in which the vector $\boldsymbol{\xi}$ is a Lagrange multiplier.

Provided that $\boldsymbol{\xi}$ is identified with the force vector, the equilibrium equations of the rod are equivalent to the condition that the equation

$$\delta\mathscr{J} = \delta\mathscr{U} - \delta\mathscr{W} + \delta\mathscr{L} = 0, \tag{28}$$

holds for all smooth vectors $\delta\boldsymbol{u}$ and $\delta\boldsymbol{q}$. When the external loads are assigned on both the ends of $\mathscr{R}$, in Eq. (28) the variation of the work done by the external loads is

$$\delta\mathscr{W} = \left[\hat{\boldsymbol{f}} \cdot \delta\boldsymbol{u} + \widehat{\boldsymbol{m}} \cdot \delta\boldsymbol{q}\right]_0^L, \tag{29}$$

where the form of the variation of the work done by the couples $\widehat{\boldsymbol{m}}$ is motivated by the form of the last term in Eq. (25). By means of integration by parts, with the use of (18) and (25), and after identification of $\boldsymbol{\xi}$ with $\boldsymbol{f}$, Eq. (28) can be written as

$$\delta\mathscr{J} = -\int_0^L \boldsymbol{f}_{,s} \cdot \delta\boldsymbol{u}\,\mathrm{d}s - \int_0^L (\boldsymbol{m}_{,s} + \boldsymbol{d}_3 \times \boldsymbol{f}) \cdot \delta\boldsymbol{q}\,\mathrm{d}s + \left[(\boldsymbol{f} - \hat{\boldsymbol{f}}) \cdot \delta\boldsymbol{u} + (\boldsymbol{m} - \widehat{\boldsymbol{m}}) \cdot \delta\boldsymbol{q}\right]_0^L = 0. \tag{30}$$

The form of this equation makes clear that $\delta\mathscr{J} = 0$ for all $\delta\boldsymbol{u}$ and $\delta\boldsymbol{q}$ if and only if the equilibrium equations (9) hold in $(0, L)$, and, on the end sections, the force and moment vectors are equal to the applied external loads $\hat{\boldsymbol{f}}$ and $\hat{\boldsymbol{m}}$. We observe that the energy functional $\mathscr{E}$ associated with a deformation of $\mathscr{R}$ is given by $\mathscr{E} = \mathscr{U} - \mathscr{W}$, and that the variation (30) of $\mathscr{J}$ coincides with the variation of the functional $\mathscr{E}$ for variations of $\boldsymbol{u}$ and $\boldsymbol{R}$ which, in consequence of (13), obey the constraint

$$\delta\boldsymbol{u}_{,s} = \delta\boldsymbol{q} \times \boldsymbol{d}_3. \tag{31}$$

Thus Eq. $(30)_2$ can be regarded as expressing the condition that in equilibrium the energy functional $\mathscr{E}$ is stationary, and the second variations of the functionals $\mathscr{E}$ and $\mathscr{J}$ coincide.

The form that the second variation $\delta^2\mathscr{J}$ assumes in correspondence of a state of equilibrium of the rod can be derived from Eq. (30) taking into account that, on an extremal, the equilibrium equations hold and thus the terms involving $\delta^2\boldsymbol{u}$ and $\delta^2\boldsymbol{q}$ vanish. With the use of (18) and (23), and after integration by parts, we find that the second variation of $\mathscr{J}$ evaluated in correspondence of an equilibrium configuration is

$$\delta^2\mathscr{J} = \int_0^L (\delta\boldsymbol{m} \cdot \delta\boldsymbol{q}_{,s} + \delta\boldsymbol{q} \times \boldsymbol{d}_3 \cdot \delta\boldsymbol{q} \times \boldsymbol{f})\,\mathrm{d}s. \tag{32}$$

The sign of the expression (32) of $\delta^2\mathscr{J}$ gives essential information on the stability of an equilibrium configuration. In particular, since a state of equilibrium in which the energy attains a minimum is regarded as stable, if $\delta^2\mathscr{J} < 0$ in an equilibrium configuration, a necessary condition for minimization is not satisfied and the equilibrium cannot be stable (cf., e.g., Antman, 1995, Chapter VII). Since, in what follows, to test the sign of $\delta^2\mathscr{J}$, we will use functions having the same form as the eigenfunctions of the linearized equilibrium problem, in next section we consider infinitesimal deformations from an arbitrary equilibrium configuration of a rod.

4. Linearization

In the deduction of the equilibrium equations for infinitesimal deformations superimposed to a finite deformation of a rod, one is concerned with three configurations: a stress-free configuration $\mathfrak{C}^{\mathrm{u}}$, an equilibrium configuration $\mathfrak{C}$ that is assumed as reference configuration, and an equilibrium configuration $\mathfrak{C}'$ that is thought of as having been obtained through a deformation starting from $\mathfrak{C}$. We denote by a "prime" the quantities pertaining to the configuration $\mathfrak{C}'$, and introduce the rotation $\boldsymbol{R}'$ that transforms the vectors $\boldsymbol{d}_i$ of $\mathfrak{C}$ into the vectors $\boldsymbol{d}'_i$ of $\mathfrak{C}'$:

$$\boldsymbol{R}' = \boldsymbol{d}'_i \otimes \boldsymbol{d}_i, \quad \boldsymbol{d}'_i = \boldsymbol{R}'\boldsymbol{d}_i, \qquad i = 1, 2, 3. \tag{33}$$

In the configuration $\mathfrak{C}'$, the strain measure $\boldsymbol{\mu}'$ can be written as

$$\boldsymbol{\mu}' = \boldsymbol{R}'\boldsymbol{\mu} + \Delta\boldsymbol{\mu}, \tag{34}$$

where $\boldsymbol{\mu}$ is the strain measure (7) in $\mathfrak{C}$ and, letting $\boldsymbol{\kappa}'$ be the curvature vector for $\mathfrak{C}'$,

$$\Delta\boldsymbol{\mu} = \boldsymbol{\kappa}' - \boldsymbol{R}'\boldsymbol{\kappa} = \frac{1}{2}\boldsymbol{R}'\boldsymbol{d}_i \times \boldsymbol{R}'_{,s}\boldsymbol{d}_i. \tag{35}$$

The moment $\boldsymbol{m}'$ in $\mathfrak{C}'$ is

$$\boldsymbol{m}' = m'_i\boldsymbol{d}'_i, \tag{36}$$

with $m'_i = m_i(\mu'_1, \mu'_2, \mu'_3; s)$. Since the rod is in equilibrium in $\mathfrak{C}'$, in this configuration we have

$$\boldsymbol{f}'_{,s} = \boldsymbol{0}, \quad \boldsymbol{m}'_{,s} + \boldsymbol{d}'_3 \times \boldsymbol{f}' = \boldsymbol{0}, \tag{37}$$

where $\boldsymbol{f}'$ is the force vector in $\mathfrak{C}'$.

We wish to find the form that Eqs. (37) assume in cases in which the state of strain and stress in the configuration $\mathfrak{C}'$ differs only slightly from the state in the configuration $\mathfrak{C}$. To this end, we observe that the rotation $\boldsymbol{R}'$ is determined by the unit vector $\boldsymbol{z}$, giving the direction of the axis of rotation of $\boldsymbol{R}'$, and the angle ϕ measuring the rotation about $\boldsymbol{z}$. Denoted by $\boldsymbol{W}$ the skew-tensor corresponding to the vector $\boldsymbol{w} = \phi\boldsymbol{z}$, the rotation $\boldsymbol{R}'$ can be written in the form $\boldsymbol{R}' = e^{\boldsymbol{W}}$, which yields

$$\boldsymbol{R}' = \boldsymbol{I} + \boldsymbol{W} + \mathrm{O}(|\boldsymbol{W}|^2). \tag{38}$$

It follows from $(33)_2$, (34), (35) and (38) that, when the vector $\boldsymbol{w}$ and its first derivative are small, at the first order the expressions of the vectors $\boldsymbol{d}_i'$, $\boldsymbol{\mu}'$, and $\Delta\boldsymbol{\mu}$ are

$$\boldsymbol{d}_i' = \boldsymbol{d}_i + \boldsymbol{w} \times \boldsymbol{d}_i, \quad i = 1, 2, 3, \tag{39}$$

$$\boldsymbol{\mu}' = \boldsymbol{\mu} + \boldsymbol{w} \times \boldsymbol{\mu} + \boldsymbol{w}_{,s}, \tag{40}$$

$$\Delta\boldsymbol{\mu} = \boldsymbol{w}_{,s}. \tag{41}$$

In order to obtain the linearized expression of the constitutive equation, we observe that for the components m_i' of $\boldsymbol{m}'$ we have

$$m_i' = m_i + \frac{\partial m_i}{\partial \mu_j}\Delta\mu_j + \mathrm{O}(|\Delta\boldsymbol{\mu}|^2), \quad i = 1, 2, 3, \tag{42}$$

where $\Delta\mu_j = \mu_j' - \mu_j$, and $|\Delta\boldsymbol{\mu}|^2 = \Delta\mu_j\Delta\mu_j$. With the use of $(33)_2$, (34), (38) and (42), Eq. (36) yields that, at the same order to within (39)–(41) hold, $\boldsymbol{m}'$ is given by

$$\boldsymbol{m}' = \boldsymbol{m} + \boldsymbol{w} \times \boldsymbol{m} + \Delta\boldsymbol{m}, \tag{43}$$

where

$$\Delta\boldsymbol{m} = \boldsymbol{C}\boldsymbol{w}_{,s}, \tag{44}$$

and $\boldsymbol{C}$ is the tensor defined by (24). In analogy with the form of the linearized constitutive equation (43), we write the force vector as

$$\boldsymbol{f}' = \boldsymbol{f} + \boldsymbol{w} \times \boldsymbol{f} + \Delta\boldsymbol{f}. \tag{45}$$

Substitution of (39), (43) and (45) into Eqs. (37), taking into account that $\mathfrak{C}$ is an equilibrium configuration, yields that, when $\boldsymbol{w}$, its first two derivatives, and $\Delta\boldsymbol{f}$ are small, the equilibrium equations for infinitesimal deformations superimposed to the finite deformation that transforms $\mathfrak{C}^{\mathrm{u}}$ into $\mathfrak{C}$, are

$$\boldsymbol{w}_{,s} \times \boldsymbol{f} + \Delta\boldsymbol{f}_{,s} = \boldsymbol{0}, \tag{46}$$

$$\boldsymbol{w}_{,s} \times \boldsymbol{m} + \Delta\boldsymbol{m}_{,s} + \boldsymbol{d}_3 \times \Delta\boldsymbol{f} = \boldsymbol{0}, \tag{47}$$

with $\Delta\boldsymbol{m}$ given by Eq. (44). Moreover, from the Eq. (39) and the condition of inextensibility (11) written for $\boldsymbol{d}_3'$, it follows that, at the first order,

$$\boldsymbol{u}_{,s} = \boldsymbol{w} \times \boldsymbol{d}_3, \tag{48}$$

where $\boldsymbol{u}$ is the displacement of the points of the axis from the axial curve $\mathscr{C}$ of the configuration $\mathfrak{C}$.

Eqs. (46)–(48), with $\Delta\boldsymbol{m}$ given by (44), constitute a system for the unknown vector functions $\boldsymbol{u}$, $\boldsymbol{w}$, and $\Delta\boldsymbol{f}$. Eq. (48), which gives the linear form of the condition that the rod axis is inextensible, implies that the displacement $\boldsymbol{u}$ is such that

$$\boldsymbol{u}_{,s} \cdot \boldsymbol{d}_3 = 0, \tag{49}$$

and that the vector $\boldsymbol{w}$ can be written as

$$\boldsymbol{w} = \boldsymbol{d}_3 \times \boldsymbol{u}_{,s} + \beta \boldsymbol{d}_3. \tag{50}$$

This last equation shows that, at each $\boldsymbol{x}(s)$, the vector $\boldsymbol{u}_{,s}(s)$ determines the two components of $\boldsymbol{w}(s)$ in the plane orthogonal to the axial curve $\mathscr{C}$. Thus, given $\boldsymbol{u}$, only one more scalar function of s (i.e., β) suffices to complete the description of $\boldsymbol{w}$. The function β appearing in Eq. (50) is the same as that introduced in Love (1944, Art. 288) and has the same meaning of being the angle through which a section is rotated about the axis in an infinitesimal deformation. In the system (46)–(48), Eq. (48) can be substituted by Eqs. (49) and (50); thus, Eqs. (46), (47), and (49), with $\boldsymbol{w}$ given by (50) and $\Delta \boldsymbol{m}$ by (44), form a system for seven scalar unknowns: the three components of $\boldsymbol{u}$, β, and the three components of $\Delta \boldsymbol{f}$.

5. Stability of annular rings with non-null twisting strain

The results of the preceding sections are now applied to a discussion on the stability of circular equilibrium configurations, in which the twisting strain μ_3 is non-null and that are obtained from materially homogeneous elastic rods that possess uniform curvature and torsion in an undistorted stress-free configuration. The rods we consider are made of a material obeying Kirchhoff's constitutive equation (15) and possess kinetic symmetry (Love, 1944, Art. 255) in the sense that in $\mathfrak{C}^{\mathrm{u}}$ the two principal moments of inertia of their sections are equal and, hence, we can put $A_1 = A_2 = A$ in Eqs. (15). For a rod of this type, an annular equilibrium configuration with $\mu_3 \neq 0$ is possible only if the axial curve in a natural state is a segment of either a cylindrical helix or straight line (Lembo, 2001), i.e., only if the rod is naturally helicoidal or naturally straight. In that case, the circular ring is obtained from the natural state by sealing the two ends of the rod after the imposition of a deformation with appropriate amounts of uniform flexural and twisting strains.

In the annular equilibrium configuration $\mathfrak{C}$, the axial curve $\mathscr{C}$ of $\mathscr{R}$ is a circle of radius R, with curvature $k = 1/R$ and torsion $\tau = 0$. Let $(\boldsymbol{e}_r, \boldsymbol{e}_z, \boldsymbol{e}_\theta)$ be the basis of a system of cylindrical coordinates (r, z, θ) for which the origin is at the center of $\mathscr{C}$, the z-axis is orthogonal to the plane containing $\mathscr{C}$, and the vector $\boldsymbol{e}_r$ has at each point of the space the direction pointing toward the z-axis. Along the axial curve $\mathscr{C}$ the basis of the cylindrical coordinate system coincides with the Frenet basis $(\boldsymbol{n}, \boldsymbol{b}, \boldsymbol{t})$ formed by the principal normal, the binormal, and the tangent. We select the origin of the coordinate θ in such a way that we have $\theta = 0$ at the point of $\mathscr{C}$ where $s = 0$, and hence $\theta = ks$. In this system of cylindrical coordinates the vectors $\boldsymbol{d}_1$, $\boldsymbol{d}_2$, and $\boldsymbol{d}_3$ have the expressions

$$\boldsymbol{d}_1 = \cos\varphi \boldsymbol{e}_r + \sin\varphi \boldsymbol{e}_z, \quad \boldsymbol{d}_2 = -\sin\varphi \boldsymbol{e}_r + \cos\varphi \boldsymbol{e}_z, \quad \boldsymbol{d}_3 = \boldsymbol{e}_\theta, \tag{51}$$

where, as in Eq. (5), φ denotes the angle from $\boldsymbol{n} = \boldsymbol{e}_r$ to $\boldsymbol{d}_1$. Introduction of the expressions (51) into Eq. (3), shows that the curvature vector for $\mathfrak{C}$ is

$$\boldsymbol{\kappa} = k(\boldsymbol{e}_z + \varphi_{,\theta} \boldsymbol{e}_\theta), \tag{52}$$

where, as we shall do in what follows, we have replaced the independent variable s by $\theta = ks$.

We firstly consider the more complex case in which in the configuration $\mathfrak{C}^{\mathrm{u}}$ the axial curve $\mathscr{C}^{\mathrm{u}}$ is a segment of an helix. Let $k^{\mathrm{u}} > 0$ and $\tau^{\mathrm{u}} \neq 0$ be the (uniform) curvature and torsion of $\mathscr{C}^{\mathrm{u}}$, and let $(\boldsymbol{n}^{\mathrm{u}}, \boldsymbol{b}^{\mathrm{u}}, \boldsymbol{t}^{\mathrm{u}})$ be the Frenet basis for this curve. Since the rod is kinetically symmetric and $\mathfrak{C}^{\mathrm{u}}$ is stress-free, at each s the triad $(\boldsymbol{d}_1^{\mathrm{u}}, \boldsymbol{d}_2^{\mathrm{u}}, \boldsymbol{d}_3^{\mathrm{u}})$ can be chosen coinciding with the Frenet basis of the helix. It follows from (3) that the curvature vector in $\mathfrak{C}^{\mathrm{u}}$ is

$$\boldsymbol{\kappa}^{\mathrm{u}} = k^{\mathrm{u}} \boldsymbol{d}_2^{\mathrm{u}} + \tau^{\mathrm{u}} \boldsymbol{d}_3^{\mathrm{u}}. \tag{53}$$

Eqs. (7), (52), and (53) yield that the strain measure in the deformation from $\mathfrak{C}^{\mathrm{u}}$ to $\mathfrak{C}$ is:

$$\boldsymbol{\mu} = k^{\mathrm{u}} \sin\varphi \boldsymbol{e}_r + (k - k^{\mathrm{u}}\cos\varphi)\boldsymbol{e}_z + (k\varphi_{,\theta} - \tau^{\mathrm{u}})\boldsymbol{e}_\theta. \tag{54}$$

It is possible to show (cf. e.g., Lembo, 2001) that in the presence of the strain (54) and with $\boldsymbol{m}$ given by Kirchhoff's constitutive equation (15), the equilibrium equations (9) can be satisfied only if φ has the values 0 or $\pm\pi$. Eqs. (51) show that the vectors $\boldsymbol{d}_1^{\mathrm{u}} = \boldsymbol{n}^{\mathrm{u}}$ and $\boldsymbol{d}_2^{\mathrm{u}} = \boldsymbol{b}^{\mathrm{u}}$ are transformed in $\boldsymbol{e}_r = \boldsymbol{n}$ and $\boldsymbol{e}_z = \boldsymbol{b}$ if $\varphi = 0$, and in $-\boldsymbol{e}_r$ and $-\boldsymbol{e}_z$ if $\varphi = \pm\pi$. We shall not consider the case $\varphi = \pm\pi$ because the corresponding configurations are known to be unstable (Tobias et al., 1996). If $\varphi = 0$, from Eqs. (15), (51), and (54) we have that in $\mathfrak{C}$ the moment is

$$\boldsymbol{m} = m_z\boldsymbol{e}_z + m_\theta\boldsymbol{e}_\theta = A(k - k^{\mathrm{u}})\boldsymbol{e}_z - C\tau^{\mathrm{u}}\boldsymbol{e}_\theta. \tag{55}$$

Then the equilibrium equations (9) require that the force vector must be

$$\boldsymbol{f} = km_\theta\boldsymbol{e}_z = -kC\tau^{\mathrm{u}}\boldsymbol{e}_z. \tag{56}$$

In components with respect to the cylindrical coordinate system, the displacement of the points of the axial curve in an infinitesimal deformation from $\mathfrak{C}$ has the expression

$$\boldsymbol{u} = u_r\boldsymbol{e}_r + u_z\boldsymbol{e}_z + u_\theta\boldsymbol{e}_\theta, \tag{57}$$

which, substituted into Eq. (50), furnishes

$$\boldsymbol{w} = w_r\boldsymbol{e}_r + w_z\boldsymbol{e}_z + w_\theta\boldsymbol{e}_\theta = -ku_{z,\theta}\boldsymbol{e}_r + k(u_{r,\theta} + u_\theta)\boldsymbol{e}_z + \beta\boldsymbol{e}_\theta. \tag{58}$$

For a kinetically symmetric rod, the strain energy σ is given by (14) with $A_1 = A_2 = A$. Accordingly, the tensor $\boldsymbol{C}$ defined by (24) is

$$\boldsymbol{C} = A(\boldsymbol{e}_r \otimes \boldsymbol{e}_r + \boldsymbol{e}_z \otimes \boldsymbol{e}_z) + C\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta, \tag{59}$$

and, in view of (58), the constitutive equation (44) becomes

$$\Delta\boldsymbol{m} = Ak((\beta - ku_{z,\theta\theta})\boldsymbol{e}_r + k(u_{r,\theta\theta} + u_{\theta,\theta})\boldsymbol{e}_z) + Ck(\beta_{,\theta} + ku_{z,\theta})\boldsymbol{e}_\theta. \tag{60}$$

We denote by Δf_r, Δf_z, and Δf_θ the components of the vector $\Delta\boldsymbol{f}$:

$$\Delta\boldsymbol{f} = \Delta f_r\boldsymbol{e}_r + \Delta f_z\boldsymbol{e}_z + \Delta f_\theta\boldsymbol{e}_\theta. \tag{61}$$

When Eqs. (55), (56), (58), (60), and (61) are introduced into the linearized equilibrium equations (46), (47) and (49), a system of seven scalar equations for the seven unknown functions β, u_r, u_z, u_θ, Δf_r, Δf_z, and Δf_θ is obtained. In order to write this system in a form more suitable for a discussion on the stability, we introduce the dimensionless quantities Ω, y, and λ, that are defined as

$$\Omega = C/A, \quad y = k^{\mathrm{u}}/k = Rk^{\mathrm{u}}, \quad \lambda = -\tau^{\mathrm{u}}/k = -R\tau^{\mathrm{u}}, \tag{62}$$

and satisfy the relations

$$\Omega > 0, \quad y > 0, \quad \lambda \neq 0. \tag{63}$$

When $\lambda(\Omega + y) \neq 0$, condition which in view of (63) always holds, the system of seven equations of the linearized equilibrium problem is equivalent (cf., e.g., Ince, 1956, Chapter VI.41) to the system:

$$\begin{aligned}&\Omega\beta_{,\theta\theta\theta\theta\theta\theta} + (y(\Omega + y - 1) + \Omega(2 + \lambda^2\Omega^2))\beta_{,\theta\theta\theta\theta} + (2y(\Omega + y - 1) + \Omega + \lambda^2\Omega^2(\Omega - y))\beta_{,\theta\theta}\\&\quad + y(\Omega + y - 1 - \lambda^2\Omega^2)\beta = 0,\end{aligned} \tag{64}$$

$$(\Omega + y)ku_{z,\theta\theta} + \Omega\beta_{,\theta\theta} - y\beta = 0, \tag{65}$$

$$\lambda\Omega k(u_\theta + u_{\theta,\theta\theta})_{,\theta\theta} + (\Omega + y)\beta_{,\theta\theta} - ku_{z,\theta\theta\theta\theta} + (\Omega + y - 1)ku_{z,\theta\theta} = 0, \tag{66}$$

$$u_{\theta,\theta} - u_r = 0, \tag{67}$$

$$\Delta f_z + Ak^3 u_{z,\theta\theta\theta} - Ak^3(\Omega + y - 1)u_{z,\theta} - Ak^2(\Omega + y)\beta_{,\theta} - \lambda C k^3 (u_{r,\theta} + u_\theta)_{,\theta} = 0, \tag{68}$$

$$\Delta f_r - \lambda C k^2 (\beta + k u_{z,\theta\theta\theta\theta}) - Ak^3 (u_\theta + u_{\theta,\theta\theta})_{,\theta\theta\theta\theta} = 0, \tag{69}$$

$$\Delta f_\theta + \Delta f_{r,\theta} - \lambda C k^2 (\beta + k u_z)_{,\theta} = 0, \tag{70}$$

in which the first equation involves only the unknown function β, the second equation involves only the functions β and u_z, and, hence, when β is known from the first equation, it contains only u_z as unknown, and so on. A function that satisfies Eq. (64) with periodic boundary conditions is of the type:

$$\beta = B\cos(n\theta + \alpha), \tag{71}$$

with n an integer and B and α constants of integration. Substitution of (71) into (64) yields (for $B \neq 0$) the relation

$$(n^2 - 1) g(y, \lambda, n) = 0, \tag{72}$$

where, for a given Ω, g is the function of y, λ, and n defined by

$$g(y, \lambda, n) = \big(\Omega n^2 - y(\Omega + y - 1)\big)(n^2 - 1) - \lambda^2 \Omega^2 (\Omega n^2 + y). \tag{73}$$

For all the values of y and λ, Eq. (72) admits the solution $n = 1$, that corresponds to the following solution of the system (64)–(70):

$$\beta = B\cos(\theta + \alpha), \tag{74}$$

$$u_r = -D\sin(\theta + \gamma), \quad u_z = -R\beta, \quad u_\theta = D\cos(\theta + \gamma), \tag{75}$$

$$\Delta f_r = 0, \quad \Delta f_z = 0, \quad \Delta f_\theta = 0, \tag{76}$$

where D and γ are constants of integration. The functions (74) and (75) describe infinitesimal rigid displacements of the rod. Precisely, the functions β and u_z describe a rigid displacement in which the circular axial curve $\mathscr{C}$ rotates about a diameter, while the functions u_r and u_θ describe a rigid displacement in which $\mathscr{C}$ undergoes a translation in the plane $z = 0$. For a given $n > 1$, Eq. (72) holds if the values of the parameters y and λ are such that

$$g(y, \lambda, n) = 0. \tag{77}$$

The pairs (y, λ) which satisfy this equation are the eigenvalues of the linearized equilibrium problem, to which the following solution of the system (64)–(70) corresponds:

$$\beta = B\cos(n\theta + \alpha), \tag{78}$$

$$u_r = \frac{\lambda\Omega}{n^2 - 1} U\beta_{,\theta}, \quad u_z = U\beta, \quad u_\theta = \frac{\lambda\Omega}{n^2 - 1} U\beta, \tag{79}$$

$$\Delta f_r = Ck^2 \lambda\beta, \quad \Delta f_z = 0, \quad \Delta f_\theta = Ck^3 U\lambda\beta_{,\theta}, \tag{80}$$

where

$$U = -\frac{R(\Omega n^2 + y)}{(\Omega + y)n^2}. \tag{81}$$

In order to evaluate the stability of the rod in the annular configuration $\mathfrak{C}$, we examine the sign of the second variation of $\mathscr{J}$, given by Eq. (32), taking as test function $\delta\boldsymbol{q}$ the vector $\boldsymbol{w}$ corresponding to functions of the form (78) and (79). With the use of $(51)_3$, (55), (56), and (58), the Eq. (32) can be written as

$$\delta^2\mathscr{J} = Ak^3\int_0^{2\pi}\{(R\beta - u_{z,\theta\theta})^2 + (u_{r,\theta\theta} + u_{\theta,\theta})^2 + \Omega(R\beta + u_{z,\theta})^2 + \lambda\Omega(u_{z,\theta}(u_{r,\theta\theta} + u_{\theta,\theta}) - (u_{r,\theta} + u_\theta)u_{z,\theta\theta}) + (1-y)(R\beta u_{z,\theta\theta} - R\beta_{,\theta}u_{z,\theta} - (u_{z,\theta})^2 - R^2\beta^2)\}\mathrm{d}\theta. \tag{82}$$

When we substitute in this equation the functions (78) and (79) for β, u_r, u_z, and u_θ, and perform the integration, we find an expression for $\delta^2\mathscr{J}$ that, after a rather lengthy computation, can be written as

$$\delta^2\mathscr{J} = \pi AB^2k\frac{\Omega n^2 + y}{(\Omega + y)^2 n^2}g(y, \lambda, n). \tag{83}$$

Thus the sign of $\delta^2\mathscr{J}$ is determined by the sign of the function g defined by (73). If we let

$$x = \lambda^2 = (-R\tau^{\mathrm{u}})^2, \tag{84}$$

Eq. (77) can be written in the form

$$-\Omega^3n^2x - \Omega^2xy + (n^2-1)(1-\Omega)y - (n^2-1)y^2 + \Omega n^2(n^2-1) = 0, \tag{85}$$

that, for each $n > 1$, represents an hyperbola Γ_n lying in the plane (x, y). Given Ω and n, we are interested in solutions x and y of (85) that, in view of the relations (63) and (84), are both positive. Thus we may put $x = py$, with $p > 0$, and from (85) we obtain the equation

$$(1 - \Omega^2p - n^2)y^2 + ((n^2-1)(1-\Omega) - \Omega^3n^2p)y + \Omega n^2(n^2-1) = 0, \tag{86}$$

whose left-hand member we may regard as a quadratic polynomial in y. An elementary analysis shows that, for each $n > 1$ and for each $p > 0$, Eq. (86) has only one positive root y_n, which corresponds to an unique pair (x_n, y_n) that satisfies the Eq. (85) of Γ_n:

$$x_n = py_n, \quad y_n = \frac{(n^2-1)(1-\Omega) - \Omega^3n^2p + \sqrt{\Delta}}{2(\Omega^2p + n^2 - 1)}, \tag{87}$$

where

$$\Delta = ((n^2-1)(1-\Omega) - \Omega^3n^2p)^2 + 4\Omega n^2(n^2-1)(\Omega^3p + n^2 - 1). \tag{88}$$

For a given n, the pairs of solutions (x_n, y_n), corresponding to different values of p, lie on that of the two branches of Γ_n that has a non-empty intersection with the region of the plane where $x > 0$ and $y > 0$. Let γ_n be the loci of the points whose coordinates (x_n, y_n) have the expressions (87), i.e., the segments of Γ_n whose points have both the coordinates positive. The curves γ_n can be represented in polar coordinates (ρ_n, ϑ) as

$$\rho_n(p) = \sqrt{(x_n)^2 + (y_n)^2}, \quad \vartheta(p) = \tan^{-1}(1/p), \tag{89}$$

where, since $p > 0$ and $n > 1$, we have $\pi/2 > \vartheta > 0$, and, by (87), $\rho_{n+1}(p) > \rho_n(p)$. In Fig. 1, with reference to a rod for which $\Omega = 2/3$, segments of the relevant branches of the hyperbolae Γ_n, for the values 2, 3, and 4 of n, are drawn.

Eqs. (72) and (83) show that, for $n > 1$, the second variation of the functional $\mathscr{J}$ vanishes on the eigencurves of the linearized system (64)–(70), whose points (λ_n, y_n) are obtained, by means of (84), from the points of the curves (89). Given an annular ring formed from a rod whose axial curve $\mathscr{C}^{\mathrm{u}}$ in a natural state is a segment of helix with curvature k^{u}, torsion τ^{u}, and length $L = 2\pi R$, let P be the point of coordinates $x = (-R\tau^{\mathrm{u}})^2 > 0$ and $y = Rk^{\mathrm{u}} > 0$, and let $\mathscr{D}$ be the open region of the plane (x, y), formed by points with

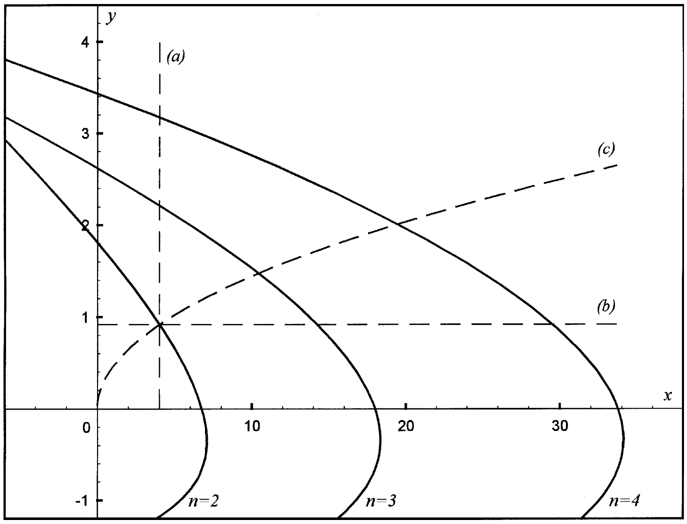

Fig. 1. The relevant segments of the hyperbolae Γ_n, for $n = 2, 3, 4$, and for $\Omega = 2/3$.

positive coordinates, that is bounded by the x-axis, the y-axis, and Γ_2. Since a bifurcation may occur at the eigenvalues of the linearized problem and an equilibrium configuration is stable when it realizes a minimum of the energy, the previous results imply that, if $P \in \mathscr{D}$, the annular configuration $\mathfrak{C}$ is stable, while if P is external to the closure of the region $\mathscr{D}$, then $g(y, \lambda, 2) < 0$, $\delta^2\mathscr{J} < 0$, and $\mathfrak{C}$ cannot be stable; thus, on the curve γ_2, where $g(y, \lambda, 2) = 0$ and $\delta^2\mathscr{J} = 0$, there is a bifurcation with loss of stability for the annular configuration. In the case of infinitesimal deformations from $\mathfrak{C}$, when k^u, τ^u, and L are such that the point P is on γ_2, the functions (78) and (79), with $n = 2$, describe the transformation of the rod into the new equilibrium configuration in which the axial curve is not planar. The branch of Γ_2 containing γ_2 intersects the axes x and y at the points x_o and y_o, with

$$x_o = 3/\Omega^2, \quad y_o = \left((1 - \Omega) + \sqrt{(1 - \Omega)^2 + 16\Omega}\right)\Big/2. \tag{90}$$

We call *critical* the value of one of the geometrical quantities k^u, $|\tau^u|$, and L, if an increment in this quantity makes unstable the annular configuration when the other two quantities are unchanged; then we have that:

(a) For given torsion τ^u and length $L = 2\pi R$ of $\mathscr{C}^u$, with $(-R\tau^u)^2 < x_o$, the coordinate y of the intersection of the curve γ_2 with the straight line $x = (-R\tau^u)^2$ (line (a) in the figure) yields the critical value $k^u = y/R$ of the curvature.
(b) For given curvature k^u and length $L = 2\pi R$ of $\mathscr{C}^u$, with $Rk^u < y_o$, the coordinate x of the intersection of the curve γ_2 with the straight line $y = Rk^u$ (line (b) in the figure) yields the critical absolute value $|\tau^u| = \sqrt{x}/R$ of the torsion.
(c) For given curvature k^u and torsion τ^u of $\mathscr{C}^u$, the coordinates of the intersection of the curve γ_2 with the parabola $y^2 = (k^u/\tau^u)^2 x$ (curve (c) in the figure) give the critical value $L = 2\pi R$ of the length, with $R = y/k^u = \sqrt{x}/|\tau^u|$.

In conclusion we briefly consider the case in which the annular equilibrium configuration has been obtained by bending a naturally straight rod with the addition of a uniform twist density κ_3. This case corresponds to the values

$$y = 0, \quad \lambda = R\kappa_3, \tag{91}$$

of the parameters. The equations governing the equilibrium in infinitesimal deformations from $\mathfrak{C}$ are obtained by putting $y = 0$ into Eqs. (64)–(70). The condition (77) for the existence of non-null (and non-rigid) solutions of these equations becomes

$$\lambda^2 \Omega^2 = n^2 - 1, \tag{92}$$

which for $n = 2$ yields $\kappa_3 = \pm\sqrt{3}/(\Omega R)$, the well-known critical value of the twist density (cf. e.g., Zajac, 1962; on the formation of loops in ropes see also Los and Ordanovich, 2002). It follows from (83) and (91) that a rod, whose axial curve in a natural state is a straight segment of length $L = 2\pi R$, is stable, when deformed in an annular ring with the addition of a uniform twist density κ_3, if

$$|\kappa_3| < \frac{\sqrt{3}}{\Omega R}. \tag{93}$$

If we put $x = (R\kappa_3)^2$, the values of x for which the annular configuration is stable when $\Omega = 2/3$, correspond in Fig. 1 to the points of the x-axis between the origin and the intersection of the x-axis with the hyperbola Γ_2.

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